Math 3235 Probability Theory

3/9/2023

Bowl with 10 balls

Some blue & some red

0.7 red

0.3 blue

N_b number of blue balls

N_r number of red balls

prior distribution for N_b

is Biononial 10, 0.3

$$\mathbb{P}(N_b > N_r) = \mathbb{P}(N_b > 5) = \sum_{i=6}^{10} \binom{10}{i} 0.3^i \, 0.7^{10-i} = 0.047$$

One blue is extracted

$$\mathbb{P}(N_b = k \mid \text{blue extracted})$$

$$\mathbb{P}(N_b = 0 \mid blue) = 0$$

$$\mathbb{P}(N_b = k \mid blue) = \frac{\mathbb{P}(blue \mid N_b = k)\,\mathbb{P}(N_b = k)}{\mathbb{P}(blue)}$$

$$\mathbb{P}(blue) = \sum_{i=1}^{10} \mathbb{P}(blue \mid N_b = i)\,\mathbb{P}(N_b = i) =$$

$$= \sum_{i=0}^{10} \frac{i}{10} \binom{10}{i} 0.3^i\, 0.7^{10-i} = 0.3$$

$$\mathbb{P}(N_b = k \mid blue) = \binom{9}{k-1} 0.3^{k-1}\, 0.7^{9-(k-1)}$$

$$\mathbb{P}(N_b = 0 \mid blue) = 0$$

$$\mathbb{P}(N_b > N_r) = \sum_{i=6}^{10} \binom{9}{i-1} 0.3^{i-1}\, 0.7^{9-(i-1)} =$$

$$0.099$$

X_i are geometric

N is geometric

$$Z = \sum_{i=1}^{N} X_i$$

$$G_{X_i}(s) = \frac{ps}{1-qs}$$

$$G_N(s) = \frac{Ps}{1-Qs}$$

$$G_Z(s) = G_N(G_{X_i}(s)) = \frac{pPs}{1-(1-Pp)s}$$

Z is geometric par pP.

$$\mathbb{P}(Z=z) = (1-Pp)^{z-1} Pp$$

Jointly distributed continuous r.v.

Chapter 6.

X, Y joint c.d.f. of X, Y

$F_{X,Y}(x, y) = \mathbb{P}(X \leq x \ \& \ Y \leq y)$

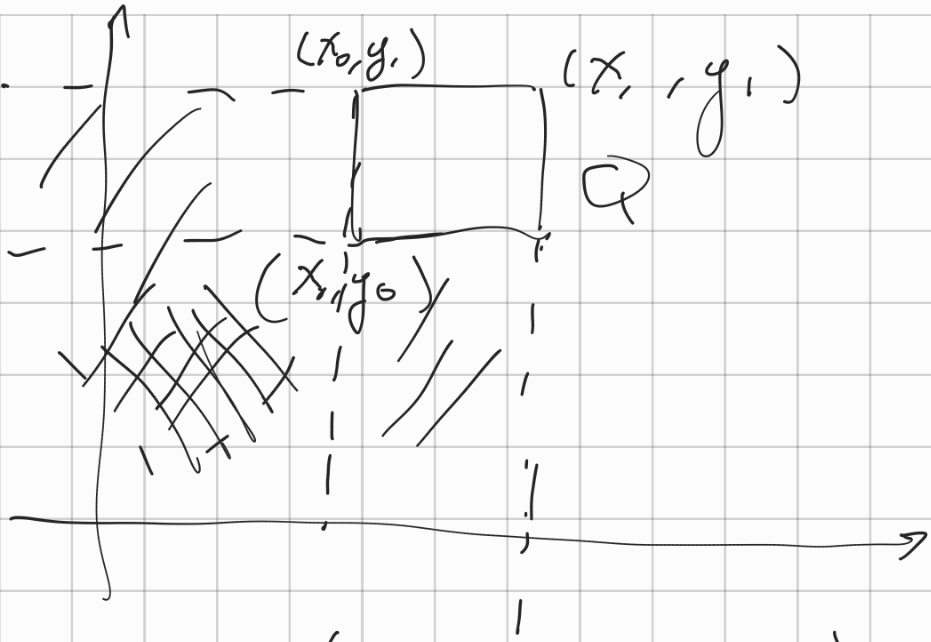
$F_{X,Y} \geqslant 0 \qquad F_{X,Y} \leq 1$

$F_{X,Y}$ is non decreasing in both x and y

$F(-\infty, -\infty) = 0 \qquad F(+\infty, +\infty) = 1$

$F_{X,Y}(x, y)$ is c.d.f. of X, Y

$F_X(x) = F_{X,Y}(x, \infty)$



$$\mathbb{P}\big((X,Y) \in Q\big) = F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0)$$

$$x_1 = x_0 + dx \qquad y_1 = y_0 + dy$$

$$Q = [x_0, x_0 + dx] \times [y_0, y_0 + dy]$$

$$\mathbb{P}\big((X,Y) \in Q\big) = F(x_0 + dx, y_0 + dy) - F(x_0, y_0 + dy) - F(x_0 + dx, y_0) + F(x_0, y_0)$$

$$F(x_0 + dx, y_0) - F(x_0, y_0) = \frac{\partial}{\partial x_0} F(x_0, y_0)\, dx$$

$$F(x_0 + dx, y_0 + dy) - F(x_0, y_0 + dy) =$$
$$\frac{\partial}{\partial x_0} F(x_0, y_0 + dy)$$

$$\mathbb{P}((X, Y) \in Q) = \frac{\partial}{\partial x_0} F(x_0, y_0 + dy)\,dx -$$
$$\frac{\partial}{\partial x_0} F(x_0, y_0)\,dx =$$
$$= \frac{\partial^2}{\partial x_0 \partial y_0} F(x_0, y_0)\,dx\,dy$$

$$\mathbb{P}(x_0 \leq X \leq x_0 + dx \;\&\; y_0 \leq Y \leq y_0 + dy) =$$
$$f(x, y)\,dx\,dy$$

$f(x,y)$ is the density or joint p.d.f. of X, Y.

We say that X, Y are

jointly distributed cont r.v.

if there exists a p.d.f.

$f(x, y)$ such that

$$\mathbb{P}\left((X, Y) \in Q \right) = \iint_Q f(x,y)\, dx\, dy$$

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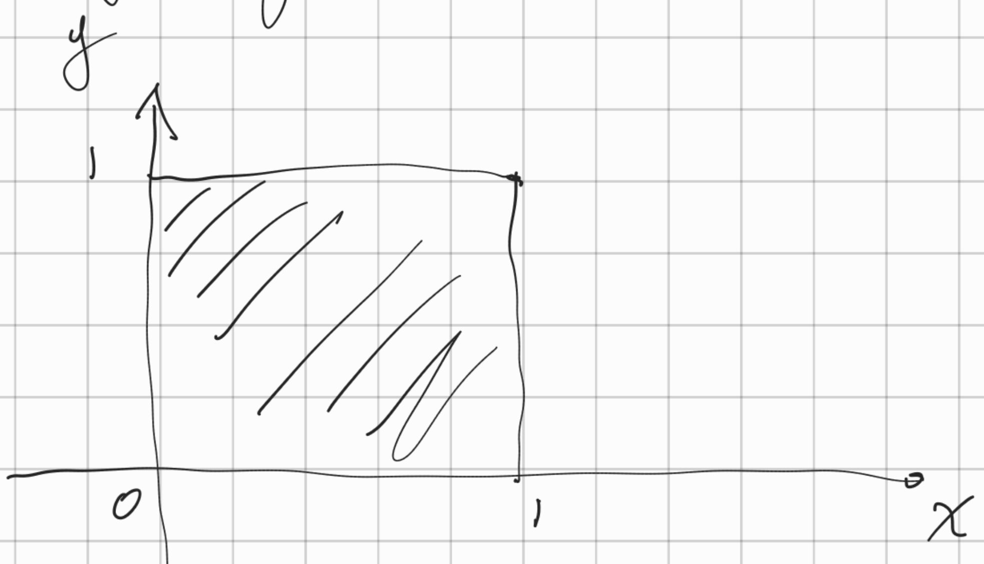
X, Y have p.d.f.

$f(x,y) = 1$ is $0 \le x, y \le 1$

$f(x,y) = 0$ otherwise.



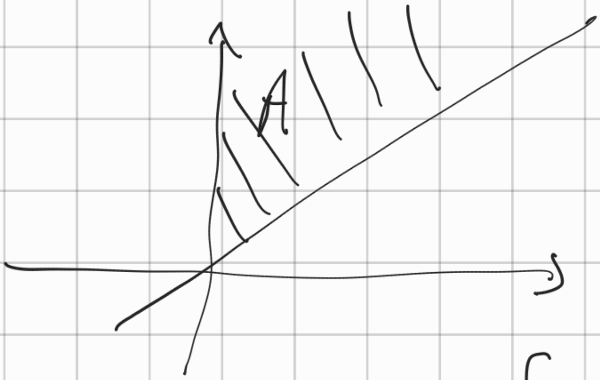
$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$$

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$$

$$\int_0^1 dx \int_0^1 dy\, 1 = 1.$$

$$\mathbb{P}(X < Y) = \mathbb{P}((X,Y) \in A)$$

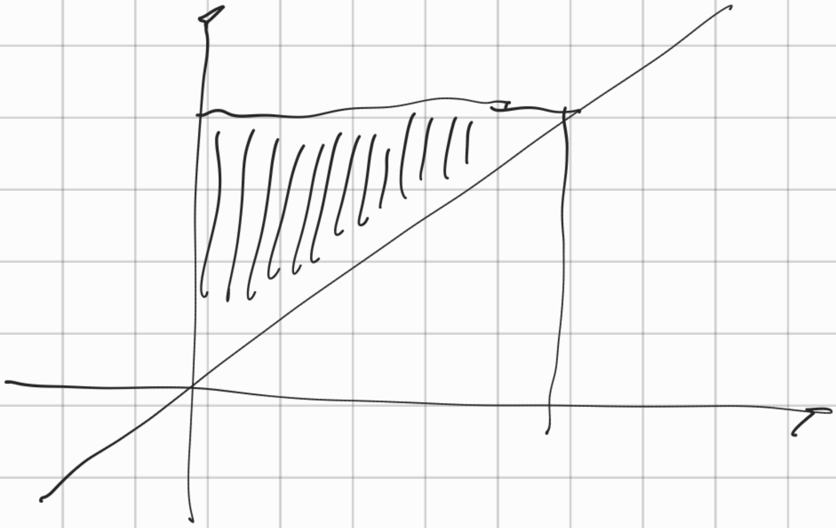
A



$$\mathbb{P}(X < Y) = \int_A f(x,y)\,dx\,dy =$$

$$\int_0^1 dx \int_x^1 dy\, 1 = \frac{1}{2}$$

$$= \int_0^1 dx\,(1-x) = -\frac{(1-x)^2}{2}\Big|_0^1 = \frac{1}{2}$$



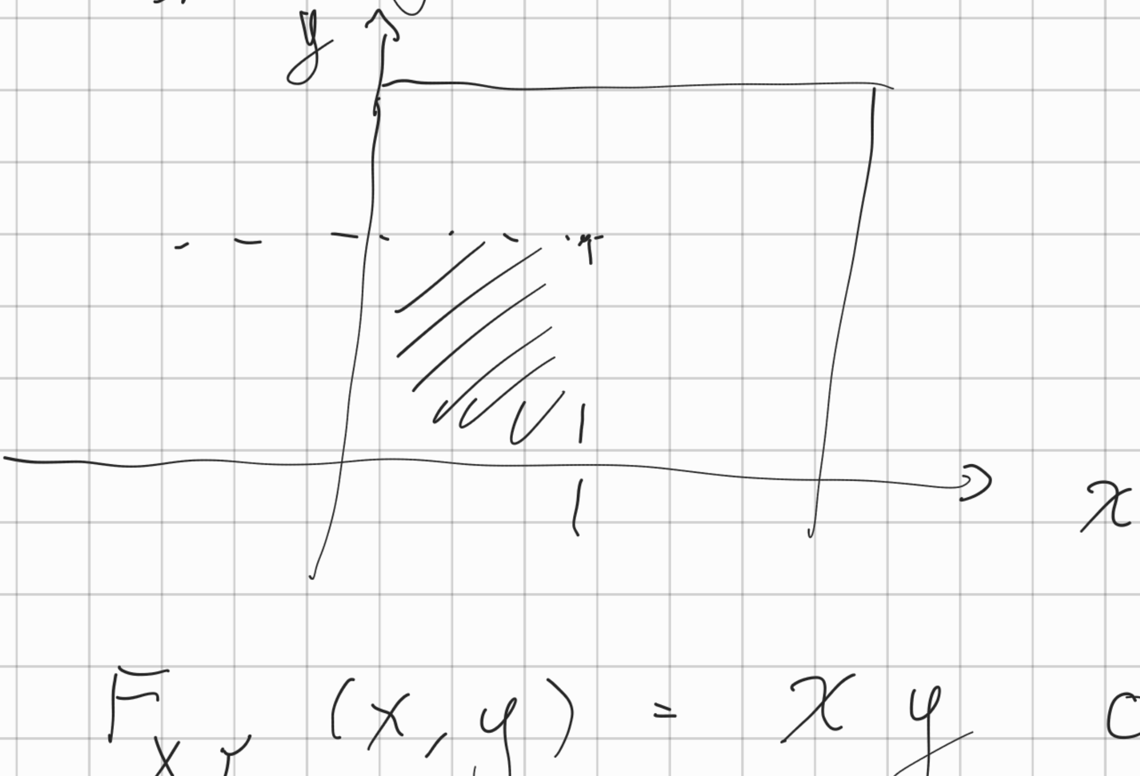
$$\mathbb{P}(X<Y) = \mathbb{P}(Y<X)$$

$$\mathbb{P}(X<Y) + \mathbb{P}(Y<X) = 1$$

$$\Downarrow$$

$$\mathbb{P}(X<Y) = \mathbb{P}(Y<X) = \frac{1}{2}$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$$



$$F_{X,Y}(x,y) = xy \qquad 0<x,y<1$$

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

$$\mathbb{P}(X \leq x \ \& \ Y \leq y) = \mathbb{P}(X \leq x) \mathbb{P}(Y \leq y)$$

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$$

X and Y are independent.